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Vibration Analysis of Heated Anisotropic Plates with Free Edge Conditions

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Numerous studies have been conducted to evaluate the effect of heating on the vibration frequencies and mode shapes of plates. Most of these studies have reported significant changes in frequency (from the room temperature frequency) only for plates with immovable edges. The present work addresses the effect of heating on vibration frequencies and mode shapes for an anisotropic plate having movable (free) edges. A Ritz formulation is used to determine the thermally induced in-plane forces and the out-of-plane vibration frequencies. Heating is applied as a steady-state temperature distribution; material properties are considered to be temperature-dependent. Tabular and plotted results are presented (as a function of lamination angle and temperature) for the first two vibration frequencies and mode shapes of a heated, single-layer Boron/epoxy laminate. The presented results demonstrate that the effect of heating on frequencies and mode shapes can be very significant for anisotropic plates with free edges, depending on the temperature distribution and the anisotropic stiffnesses.

Introduction

THE design of hypersonic and high-speed vehicles (such as the proposed National Aerospace Plane) presents a number of technological challenges. For structural designers, one major challenge is extreme temperature combined with high structural loads. New material systems and structural concepts must be developed for this harsh environment. To evaluate proposed designs, the effects of elevated temperature must be well documented, analytically and experimentally. This study addresses one aspect of the high-temperature problem: the effect of heating on a structure's elastic vibration characteristics.

It is well known that heating changes the elastic behavior of a structure in two ways: the material properties change and thermally induced stresses can develop (depending on the temperature distribution and boundary conditions). Several studies[1–8] have investigated the effect of these two changes on elastic vibration frequencies and mode shapes. Tang[2] studied the vibration of simply supported, rectangular, titanium alloy plates with temperature-dependent properties and movable in-plane edges. Three special cases were considered: 1) a plate with temperature varying linearly through the thickness only, 2) a plate with temperature varying linearly over the surface and constant through the thickness, and 3) a plate with temperature varying linearly with all three coordinates. For all three cases, the change in fundamental frequency (from the room temperature frequency) was less than 10%, even for temperatures as high as 600°F. Snyder and Kehoe[7] experimentally and analytically studied several cases of uniform and nonuniform heating to determine the effect of in-plane forces and temperature-dependent material properties on the vibration frequencies of aluminum plates with free edge conditions. For all cases studied, the largest experimentally measured change in frequency was 10%. Locke[8] analytically studied several of the cases reported in Ref. 7 and reported similar results.

For plates with restrained in-plane edges, the effect of heating is much more significant. As demonstrated by Refs. 1 and 4, relatively small changes in temperature produce significant changes in frequency due to the development of compressive in-plane forces. And when the in-plane forces are sufficient to produce buckling, one of the vibration frequencies tends to zero. Bailey[3,5] reported this same type of behavior for plates with various boundary conditions and temperature distributions, including plates with movable in-plane edges. Bailey also found that as temperature is increased, the first and second vibration modes can change positions in the frequency spectrum.

All of the above-mentioned studies are for isotropic plates. Tomar and Gupta[6] studied the effect of a constant thermal gradient on the free axisymmetric vibrations of an orthotropic elastic rectangular plate of linearly varying thickness with temperature-dependent material properties and movable in-plane edges. In-plane force effects were not included. Their results also indicate a small change in the elastic vibration frequencies. With the exception of Bailey, all of the studies conducted for plates with movable in-plane edges have reported small changes in elastic vibration frequencies due to heating. The key difference between Bailey's studies and Refs. 2 and 6–8 is the development of significant in-plane forces. Bailey studied temperature distributions that produced significant in-plane forces, even for plates with movable edges. Consequently, the vibration frequencies changed by a considerable amount.

The present study substantiates the significant effect of temperature distributions (and in-plane forces) on vibration frequencies and mode shapes by considering two nonuniform temperature distributions that both have the same maximum temperature. New results are also presented to demonstrate the significant effect of material anisotropy.

Mathematical Formulation

This section presents the governing equations, both classical differential equations and their variational equivalents, for a heated anisotropic rectangular plate (Fig. 1). Material ani-

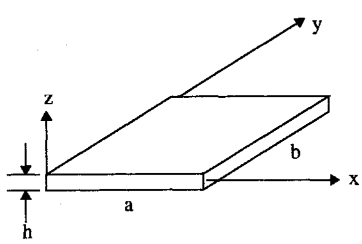

Fig. 1 Plate geometry.

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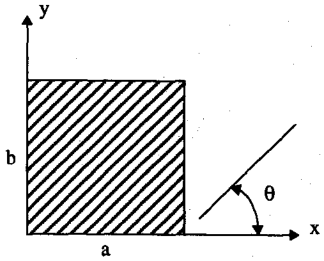

Fig. 2 Laminate geometry.

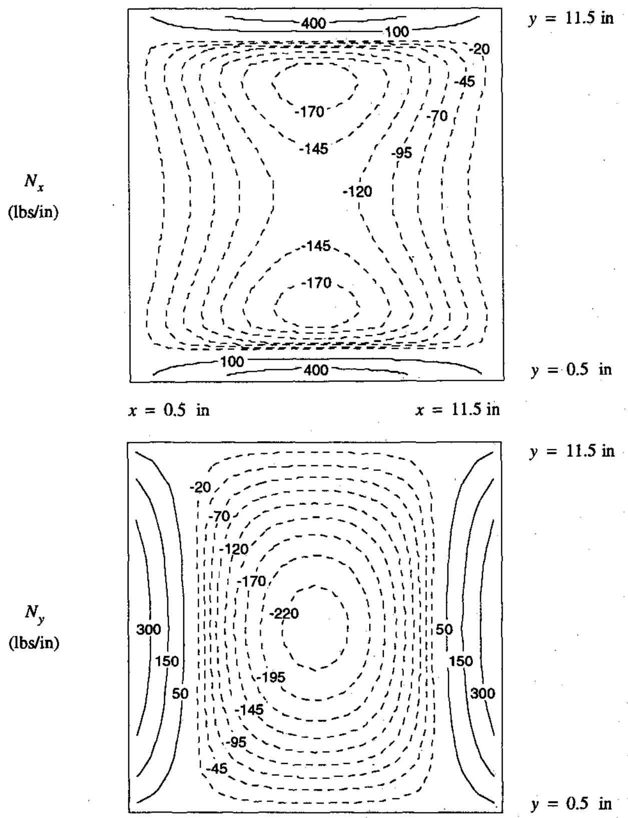

Fig. 3 Case 1: in-plane forces at ΔT_{max} = 300°F and θ = 0 deg (constant material properties).

sotropy is accounted for by considering a single generally orthotropic layer oriented at an angle of θ with respect to the x-coordinate direction (Fig. 2). The plate is assumed to be thin, flat, and initially stress-free with no applied mechanical loads. Material properties are assumed to be temperature-dependent, and the temperature distribution is assumed to be a function of x and y only (constant through the thickness).

A Ritz method is developed to compute the thermally induced in-plane forces and the resulting out-of-plane vibration frequencies. The method is formulated in terms of the out-of-plane displacement and the in-plane stress function. Both the displacement and the stress function are expressed as simple algebraic polynomials modified to satisfy the essential boundary conditions. Baharlou and Leissa[9] used a similar displacement-based Ritz approach (in-plane displacements were used instead of a stress function) to investigate the buckling and vibration of generally laminated plates with various edge conditions. Their results demonstrate that the method is efficient and accurate for the vibration of nonheated plates. To verify the efficiency and accuracy for heated plates, the thermally induced in-plane forces are compared with Fourier series analytical results, and the vibration frequencies are compared with experimental and finite element results.[7,8] The Fourier series analysis is based on an approach originally developed by Green[10] for isotropic plates. Whitney[11,12] modified Green's method for the analysis of anisotropic plates.

Governing Equations

The classical differential equations describing the free vibration behavior of a heated plate with no temperature variation through the thickness are given by Ref. 13:

$$N_{x,x} + N_{xy,y} = 0 \qquad N_{xy,x} + N_{y,y} = 0$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} - \rho h w_{,tt} = 0 \tag{1}$$

where ρ is the mass density and h is the plate thickness. Using classical lamination theory,[13] the in-plane forces $\boldsymbol{N}$ and moments $\boldsymbol{M}$ can be related to the in-plane strains $\boldsymbol{e}$ and bending curvatures $\boldsymbol{\kappa}$ as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} - \begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

in which

$$\begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix} \qquad \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = - \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

$$N_i^T = \Delta T \int_{-h/2}^{h/2} \left(\sum_j Q_{ij} \alpha_j \right) \mathrm{d}z$$

$$(A_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z^2) \,\mathrm{d}z \tag{2}$$

where u and v are the in-plane displacements, w is the out-of-plane displacement, N_i^T are the thermal forces, ΔT is the temperature distribution, Q_{ij} are the transformed reduced stiffnesses, α_j are the transformed coefficients of thermal expansion, and (A_{ij}, D_{ij}) are the laminate stiffnesses. Substituting the moments from Eqs. (2) into the last of Eqs. (1) yields

$$\begin{aligned} &-(D_{11} w_{,xx} + D_{12} w_{,yy} + 2D_{16} w_{,xy})_{,xx} \\ &- 2(D_{16} w_{,xx} + D_{26} w_{,yy} + 2D_{66} w_{,xy})_{,xy} \\ &- (D_{12} w_{,xx} + D_{22} w_{,yy} + 2D_{26} w_{,xy})_{,yy} \\ &+ N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} - \rho h w_{,tt} = 0 \end{aligned} \tag{3}$$

Equation (3) is the governing out-of-plane differential equation for a heated anisotropic plate with temperature-dependent material properties in terms of w. Similar displacement-based equations can be obtained for the in-plane behavior by substituting the in-plane forces from Eq. (2) into the first two of Eq. (1).

An alternative in-plane differential equation can be obtained using a stress function formulation[14]

$$(N_x \quad N_y \quad N_{xy}) = (F_{,yy} \quad F_{,xx} \quad -F_{,xy}) \tag{4}$$

where the stress function F must satisfy the compatibility equation

$$e_{x,yy} + e_{y,xx} - e_{xy,xy} = 0 \tag{5}$$

From Eq. (2) the in-plane strains can be expressed as[13]

$$\boldsymbol{e} = \boldsymbol{A}^{-1}(\boldsymbol{N} + \boldsymbol{N}^T) = \boldsymbol{a}(\boldsymbol{N} + \boldsymbol{N}^T) \tag{6}$$

Table 1 Case 1: convergence of maximum in-plane compressive forces at ΔT_{max} = 300°F (constant material properties)

	θ = 0 deg		θ = 30 deg		θ = 60 deg		θ = 90 deg	
nf	N_x	N_y	N_x	N_y	N_x	N_y	N_x	N_y
49	−257.3[a]	−225.8	−203.9	−209.8	−101.0	−98.2	−39.9	−45.4
64	−257.3	−225.8	−197.6	−208.2	−100.3	−95.2	−39.9	−45.4
81	−202.9	−233.4	−182.7	−205.3	−98.9	−88.0	−41.2	−35.8
100	−202.9	−233.4	−178.5	−203.7	−98.1	−86.0	−41.2	−35.8
121	−178.1	−232.8	−175.5	−202.0	−97.2	−84.5	−41.1	−31.5
144	−178.1	−232.8	−175.5	−201.6	−97.0	−84.8	−41.1	−31.5
169	−185.5	−232.9	−175.5	−201.5	−97.4	−85.1	−41.2	−34.7
Fourier series solution	−184.6	−233.3	−174.9	−202.1	−97.3	−84.3	−41.2	−32.6

[a]lbs/in.

Table 2 Case 1: convergence of vibration frequencies at ΔT_{max} = 300°F

	θ = 0 deg		θ = 30 deg		θ = 60 deg		θ = 90 deg	
nw	f_1	f_2	f_1	f_2	f_1	f_2	f_1	f_2
				nf = 121				
49	44.1[a]	113.9	57.7	128.8	64.3	95.4	60.0	72.8
64	44.1	113.9	57.3	128.5	64.2	95.3	60.0	72.8
				nf = 144				
49	44.1	113.9	57.6	128.8	64.3	95.4	60.0	72.8
64	44.1	113.9	57.3	128.5	64.2	95.3	60.0	72.8
				nf = 169				
64	44.1	113.9	57.3	128.5	64.2	95.3	60.0	72.8

[a]Hz.

Substitution of Eqs. (4) and (6) into Eq. (5) yields

$$\begin{aligned}&[a_{11}(F_{,yy} + N_x^T) + a_{12}(F_{,xx} + N_y^T)\\&+ a_{16}(-F_{,xy} + N_{xy}^T)]_{,yy} + [a_{12}(F_{,yy} + N_x^T)\\&+ a_{22}(F_{,xx} + N_y^T) + a_{26}(-F_{,xy} + N_{xy}^T)]_{,xx}\\&- [a_{16}(F_{,yy} + N_x^T) + a_{26}(F_{,xx} + N_y^T)\\&+ a_{66}(-F_{,xy} + N_{xy}^T)]_{,xy} = 0\end{aligned} \tag{7}$$

Equation (7) is the governing in-plane differential equation for a heated anisotropic plate with temperature-dependent material properties in terms of the stress function F. The advantage in using Eq. (7), instead of a displacement-based approach, is that there is only one unknown and one equation instead of two unknowns and two equations. But Eq. (7) is also a higher order equation than the displacement-based, in-plane equations. The choice is usually dictated by the boundary conditions.

For the case of constant material properties, the inverted stiffnesses a_{ij} are constant, and N_i^T are functions of ΔT. Applying these conditions to Eq. (7) yields

$$\begin{aligned}&a_{22}F_{,xxxx} - 2a_{26}F_{,xxxy} + (2a_{12} + a_{66})F_{,xxyy}\\&- 2a_{16}F_{,xyyy} + a_{11}F_{,yyyy} = q\end{aligned}$$

where

$$\begin{aligned}q = &-a_{11}N^T_{x,yy} - a_{12}(N^T_{y,yy} + N^T_{x,xx})\\&- a_{16}(N^T_{xy,yy} - N^T_{x,xy}) - a_{22}N^T_{y,xx}\\&- a_{26}(N^T_{xy,xx} - N^T_{y,xy}) + a_{66}N^T_{xy,xy}\end{aligned} \tag{8}$$

Variational equivalents to Eqs. (3) and (7) can be obtained using the principle of virtual work for Eq. (3) and the principle of complementary virtual work for Eq. (7). The resulting variational forms of Eqs. (3) and (7), respectively, are given by

$$\begin{aligned}\int [(&\delta\kappa_x M_x + \delta\kappa_y M_y + \delta\kappa_{xy} M_{xy}) + \delta w_{,x}(N_x w_{,x} + N_{xy} w_{,y})\\&+ \delta w_{,y}(N_y w_{,y} + N_{xy} w_{,x}) + \delta w(\rho h w_{,tt})]\, \mathrm{d}A = 0\end{aligned} \tag{9}$$

$$\int (\delta N_x e_x + \delta N_y e_y + \delta N_{xy} e_{xy})\, \mathrm{d}A = 0 \tag{10}$$

where the moments and bending curvatures are given by Eqs. (2), and the in-plane forces and strains are given by Eqs. (4) and (6).

The boundary conditions on a free edge require that all forces and moments vanish. In terms of normal and tangential components, n and s, respectively, these conditions can be expressed as[14]

$$\frac{\partial M_s}{\partial s} + Q_n = 0 \qquad M_n = 0 \tag{11}$$

$$N_s = 0 \qquad N_n = 0 \tag{12}$$

where M_s and M_n are the twisting and normal components of the edge moment resultant, Q_n is the transverse shear stress resultant along an edge, and N_s and N_n are the shear and normal components of the edge in-plane force resultant. For

Table 3 Case 2: convergence of vibration frequencies at ΔT_{max} = 300°F

	θ = 0 deg		θ = 30 deg		θ = 60 deg		θ = 90 deg	
nw	f_1	f_2	f_1	f_2	f_1	f_2	f_1	f_2
				nf = 121				
49	13.0[a]	91.5	42.1	86.1	59.3	92.8	51.3	102.9
64	13.0	91.5	42.1	86.1	59.3	92.7	51.3	102.9
				nf = 144				
49	13.0	91.5	42.1	86.1	59.3	92.8	51.3	102.9
64	13.0	91.5	42.1	86.1	59.3	92.7	51.3	102.9
				nf = 169				
64	13.0	91.5	42.1	86.1	59.3	92.7	51.3	102.9

[a]Hz.

Table 4 Convergence of finite element fundamental frequency at ΔT_{max} = 300°F and θ = 0 deg

	Number of elements				
	9	16	25	36	49
Case 1	44.5[a]	44.0	43.9	43.9	43.9
Case 2	12.0	12.7	12.8	12.9	12.9

[a]Hz.

a stress function formulation, Eq. (12) is equivalent to the following[14]:

$$\frac{\partial F}{\partial n} = 0 \qquad F = 0 \tag{13}$$

The out-of-plane boundary conditions given by Eq. (11) are natural, and the in-plane boundary conditions given by Eq. (13) are essential.

Method of Solution

No exact solutions to Eqs. (3) and (7) with freely supported boundary conditions [Eqs. (11) and (13)] have been reported. For the case of constant material properties, Eq. (7) reduces to Eq. (8). The boundary conditions given by Eq. (13) taken with Eq. (8) are mathematically equivalent to the bending analysis of a clamped anisotropic plate subjected to a distributed loading. A Fourier series solution can be obtained using the method of Ref. 11. Enforcing the boundary conditions results in an infinite system of simultaneous equations that can be truncated to obtain the desired degree of accuracy. The Fourier series solution is exact in the sense that the differential equation and the boundary conditions are both satisfied. The method presented in Ref. 11 can only be applied to the solution of equations with constant coefficients and does not apply to Eq. (7), since a_{ij} (which are temperature-dependent), are functions of x and y.

Approximate solutions to Eqs. (3) and (7) can be obtained from the variational forms given by Eqs. (9) and (10). Using the Ritz method, the out-of-plane deflection and the in-plane stress function are expressed as simple algebraic polynomials:

$$w(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} W_{mn} \zeta^m \eta^n = \boldsymbol{HW}$$

$$F(x, y) = \sum_{i=0}^{I} \sum_{j=0}^{J} F_{ij} \zeta^i \eta^j = \boldsymbol{LF}$$

where

$$\zeta = \frac{2x}{a} - 1 \qquad \eta = \frac{2y}{b} - 1 \tag{14}$$

From Eqs. (2), (4), and (14), the bending curvatures, bending slopes, and in-plane forces can be expressed as

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{H}_{,xx} \\ -\boldsymbol{H}_{,yy} \\ -2\boldsymbol{H}_{,xy} \end{bmatrix} \boldsymbol{W} = \boldsymbol{B}_w \boldsymbol{W}$$

$$\begin{bmatrix} w_{,x} \\ w_{,y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{,x} \\ \boldsymbol{H}_{,y} \end{bmatrix} \boldsymbol{W} = \boldsymbol{B}_g \boldsymbol{W}$$

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{,yy} \\ \boldsymbol{L}_{,xx} \\ -\boldsymbol{L}_{,xy} \end{bmatrix} \boldsymbol{F} = \boldsymbol{B}_f \boldsymbol{F} \tag{15}$$

Substituting Eqs. (14) and (15), the moments from Eqs. (2), and the in-plane strains from Eq. (6) into Eqs. (9) and (10), produces equations of the form

$$\boldsymbol{K}_f \boldsymbol{F} = \boldsymbol{P}_f \qquad [\boldsymbol{K}_w + \boldsymbol{K}_g]\boldsymbol{W} + \boldsymbol{M}_w \boldsymbol{W}_{,tt} = \boldsymbol{0}$$

where

$$\boldsymbol{K}_f = \int \boldsymbol{B}_f^T \boldsymbol{a} \boldsymbol{B}_f \, \mathrm{d}A \qquad \boldsymbol{P}_f = -\int \boldsymbol{B}_f^T \boldsymbol{a} \boldsymbol{N}^T \, \mathrm{d}A$$

$$\boldsymbol{K}_w = \int \boldsymbol{B}_w^T \boldsymbol{D} \boldsymbol{B}_w \, \mathrm{d}A \qquad \boldsymbol{K}_g = \int \boldsymbol{B}_g^T \boldsymbol{n} \boldsymbol{B}_g \, \mathrm{d}A$$

$$\boldsymbol{M}_w = \int \boldsymbol{H}^T \rho h \boldsymbol{H} \, \mathrm{d}A \qquad \boldsymbol{n} = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \tag{16}$$

where the first equation corresponds to the in-plane equation, Eq. (10), and the second equation corresponds to the out-of-plane equation, Eq. (9). Since the Ritz method requires that the essential boundary conditions be satisfied, the stress function boundary conditions for the in-plane forces, Eq. (13), must be enforced. Following the approach of Ref. 9, Eq. (13) can be enforced by applying constraint equations

$$\boldsymbol{F} = \boldsymbol{C}\boldsymbol{f} \tag{17}$$

where $\boldsymbol{C}$ is a constraint matrix and $\boldsymbol{f}$ is a reduced vector. Also, for free vibration

$$\boldsymbol{W}_{,tt} = -\omega^2 \boldsymbol{W} \tag{18}$$

where ω is the natural frequency. Substituting Eqs. (17) and (18) into Eq. (16) results in the following equations:

$$\boldsymbol{k}_f \boldsymbol{f} = \boldsymbol{p}_f \qquad [\boldsymbol{K}_w + \boldsymbol{K}_g]\boldsymbol{W} = \omega^2 \boldsymbol{M}_w \boldsymbol{W}$$

where

$$\boldsymbol{k}_f = \boldsymbol{C}^T \boldsymbol{K}_f \boldsymbol{C} \qquad \boldsymbol{p}_f = \boldsymbol{C}^T \boldsymbol{P}_f \tag{19}$$

Table 5 Heated aluminum plate vibration frequencies

	Uniform heating, 400°F			Nonuniform heating, 400, 300, 200°F		
	Experimental[7]	Finite element[7]	Ritz	Experimental[7]	Finite element[7]	Ritz
f_1	13.6[a]	12.5	13.1	13.9	13.2	13.5
f_2	34.3	30.5	31.9	34.7	31.5	32.6
f_3	39.3	39.3	36.1	40.0	39.0	37.0
f_4	69.9	64.5	65.6	71.1	66.9	66.8

[a]Hz.

Table 6 Case 1: effect of heating on vibration frequencies

	$\theta = 0$ deg		$\theta = 30$ deg		$\theta = 60$ deg		$\theta = 90$ deg	
ΔT_{max}	f_1	f_2	f_1	f_2	f_1	f_2	f_1	f_2
0.0[a]	65.0[b]	114.6	80.5	107.5	80.5	107.5	65.0	114.6
50.0	80.9	109.6	86.8	112.3	82.9	106.0	65.8	108.9
100.0	92.5	101.3	85.7	118.7	82.6	104.9	65.9	102.9
150.0	90.2	101.1	80.8	124.2	80.0	103.9	65.5	96.3
200.0	76.7	107.4	73.8	127.9	75.7	102.3	64.5	89.2
250.0	61.2	111.7	65.7	129.5	70.4	99.7	62.7	81.4
300.0	44.1	113.9	57.3	128.5	64.2	95.3	60.0	72.8

[a]°F. [b]Hz.

Table 7 Case 2: effect of heating on vibration frequencies

	$\theta = 0$ deg		$\theta = 30$ deg		$\theta = 60$ deg		$\theta = 90$ deg	
ΔT_{max}	f_1	f_2	f_1	f_2	f_1	f_2	f_1	f_2
0.0[a]	65.0[b]	114.6	80.5	107.5	80.5	107.5	65.0	114.6
50.0	58.2	112.1	75.1	103.8	77.2	105.0	62.8	112.8
100.0	51.0	109.3	69.2	100.4	73.8	102.6	60.6	111.0
150.0	43.4	106.0	62.9	97.0	70.3	100.2	58.4	109.1
200.0	35.1	102.1	56.1	93.5	66.6	97.7	56.1	107.2
250.0	25.6	97.4	49.2	90.0	63.0	95.3	53.7	105.1
300.0	13.0	91.5	42.1	86.1	59.3	92.7	51.3	102.9

[a]°F. [b]Hz.

where the first equation is used to determine the in-plane forces due to heating, and the second equation is used to determine the out-of-plane vibration frequencies and corresponding mode shapes.

Numerical Results

Except for Table 5, all results are for a $12 \times 12 \times 0.12$ (dimensions in inches) single-layer Boron/epoxy plate subjected to the following temperature distributions:

Case 1:

$$T(x) = 4\Delta T_{max}(1 - x/a)(x/a) + T_0$$

Case 2:

$$T(x) = \Delta T_{max}(x/a)^2 + T_0$$

where $T_0 = 75°F$.

Material properties as a function of temperature T are based on the data given in Ref. 15 for $T > 50°F$:

$$E_1 = 30 \times 10^6 \text{ psi} \quad E_2 = (3.06 \times 10^6 - 5.63 \times 10^3 T) \text{ psi}$$

$$\nu_{12} = 0.21 \quad G_{12} = (0.921 \times 10^6 - 1.83 \times 10^3 T) \text{ psi}$$

$$\alpha_1 = 2.42 \times 10^{-6} \text{ in./in./°F} \quad \alpha_2 = 13.7 \times 10^{-6} \text{ in./in./°F}$$

A value of 1.475×10^{-4} lb-s^2/in.^{4} was used for ρ.

In-Plane Forces

For problems of the type considered herein, accurate computation of the thermally induced in-plane forces can be quite difficult. Consider, for example, the above-defined case 1 temperature distribution. Fourier series results (for a lamination angle of $\theta = 0$ deg and $\Delta T_{max} = 300°F$) are shown in Fig. 3. As demonstrated, the in-plane forces vary substantially over the surface of the plate, and sharp gradients occur near the plate edges. Consequently, the Ritz method is slow to converge. Table 1 presents a convergence study for the maximum in-plane compressive forces (as a function of lamination angle) at $\Delta T_{max} = 300°F$. These results (and the results shown in Fig. 3) are based on using 49 (7×7) terms for the Fourier series. Since the Fourier series method is valid only for constant material properties, room temperature properties ($T = 75°F$) were used for all analyses. Ritz results are shown for the number of terms, $nf = (I + 1) \times (J + 1)$, in the stress function polynomial series. As demonstrated, the 121, 144, and 169 term Ritz solutions are in excellent agreement with the Fourier series results.

Vibration Frequencies and Mode Shapes

Determination of the vibration frequencies is a two-step process: 1) compute the in-plane forces due to heating, and 2) compute the vibration frequencies. Tables 2 and 3 present frequency convergence studies for the case 1 and 2 temperature distributions at the maximum temperature ($\Delta T_{max} = 300°F$), where the number of terms in the out-of-plane deflection polynomial series is $nw = (M + 1) \times (N + 1)$. Based on these results, all other analyses were conducted using 121 terms ($nf = 121$) for the stress function and 64 terms ($nw = 64$) for the out-of-plane deflection.

The Ritz method was chosen based upon the efficiency and accuracy demonstrated by the results given in Ref. 9. Tables 1–3 substantiate that the method is also well suited for the present problem. To further demonstrate the efficiency and accuracy of the present method, comparisons are made with experimental and finite element resuts.[7,8] Finite element results for the fundamental frequency (at a lamination angle of $\theta = 0$ deg and $\Delta T_{max} = 300°F$) are shown in Table 4. These

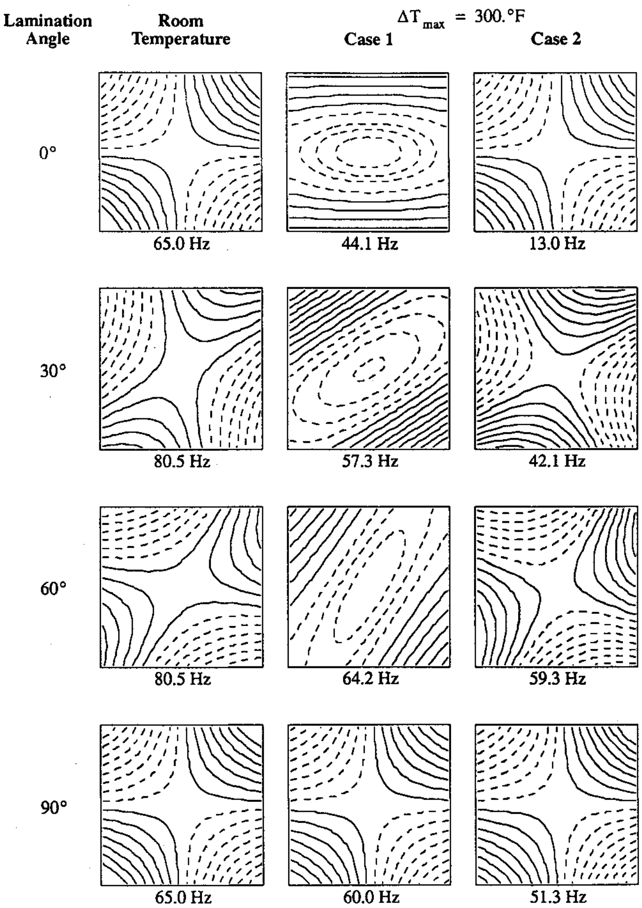

Fig. 4 Effect of heating on first mode shape.

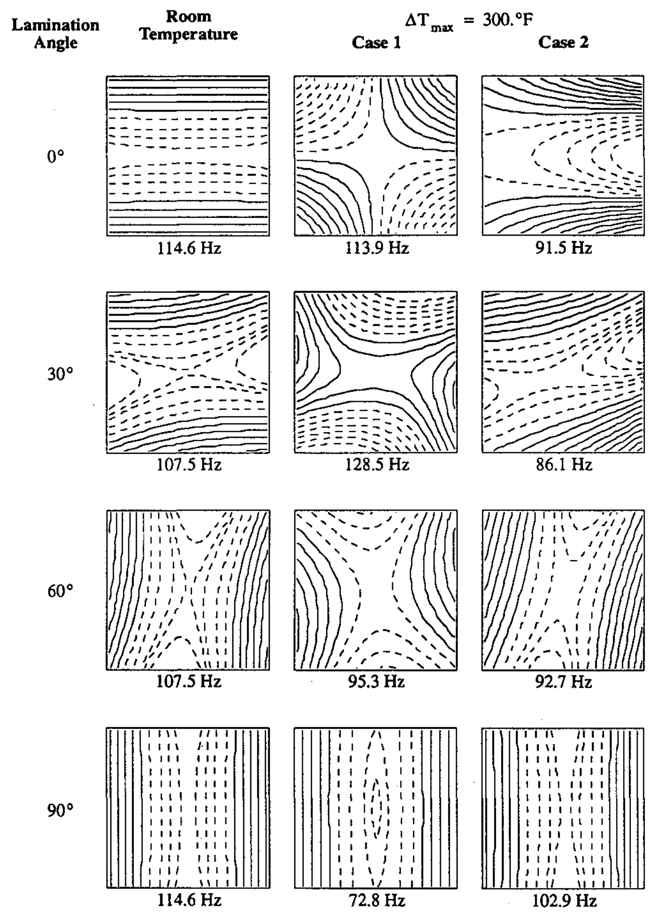

Fig. 5 Effect of heating on second mode shape.

results were obtained using the 48 degree-of-freedom (12 DOF per node) rectangular plate finite element given in Ref. 8. Results are shown for 9, 16, 25, 36, and 49 equal-sized elements with total DOF of 189, 297, 429, 585, and 765, respectively. For the case 2 temperature distribution, 36 elements with 585 DOF are required to achieve convergence compared with 185 DOF ($nf + nw = 185$) for the Ritz solution. The results shown in Table 5 are for the 50 × 12 × 0.19 (in.) heated aluminum plate of Ref. 7.

The effect of heating on the first two vibration frequencies is shown in Table 6 for the case 1 temperature distribution and Table 7 for the case 2 temperature distribution. Results for the mode shapes at room temperature ($\Delta T_{max} = 0$) and the maximum temperature ($\Delta T_{max} = 300°F$) are presented in Figs. 4 and 5. These results demonstrate that the effect of heating on frequencies and mode shapes can be very significant, depending on the temperature distribution and the lamination angle. At a lamination angle of $\theta = 0$ deg, the case 2 fundamental frequency changes from 65.0 Hz at room temperature to 13.0 Hz at the maximum temperature, while the fundamental mode shape does not change. The case 1 temperature distribution does not have such a significant impact on the frequency, but the fundamental mode shape at $\theta = 0$ deg changes from a torsional mode at room temperature to a bending mode at the maximum temperature, and the second mode shape changes from a bending mode at room temperature to a torsional mode at the maximum temperature. At a lamination angle of $\theta = 90$ deg, the first mode shape is always torsional and the second is always bending. For the intermediate angles of 30 and 60 deg, the room temperature and case 2 mode shapes are very similar, while the case 1 mode shape is always different. These mode shape changes are consistent with the isotropic plate results reported by Bailey[3,5]: for some cases the first and second vibration mode shapes change positions in the frequency spectrum.

It is interesting to note that a substantial change in frequency is not necessarily accompanied by a change in mode shape. In fact, the most severe change in frequency (case 2 at $\theta = 30$ deg) occurs when the mode shape remains unchanged. The converse is also true: a change in mode shape is not necessarily accompanied by a substantial change in frequency. Also note that the second frequency (for case 1 at $\theta = 30$ deg) increases from the room temperature value.

Conclusions

A Ritz formulation has been used to determine the effect of heating on vibration frequencies and mode shapes for an anisotropic plate having free edges. Determination of the vibration frequencies is a two-step process: 1) compute the inplane forces due to heating, and 2) compute the vibration frequencies. For problems of the type considered herein, the first step is the most difficult. The presented results demonstrate that the effect of heating can be very significant, depending on the temperature distribution and the anisotropic stiffnesses.

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